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Lattice-based Sum Construction of Nullnorms on Bounded Lattices

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ABSTRACT

Nullnorms on bounded lattices are generalizations of t-norms and t-conorms with a zero element laying anywhere in the underlying lattices. In this paper, new methods for constructing nullnorms on bounded lattices are proposed. The proposed construction methods are based on the lattice-based sum of lattices that has been recently introduced by El-Zekey et al. (see [9]), for building new (bounded) lattices from fixed ones indexed by a (finite) lattice-ordered index set. Subsequently, the new construction methods are applied for building several new families of nullnorms on bounded lattices. As a by-product, lattice-based sum constructions of t-norms and t-conorms on bounded lattices have been obtained. Furthermore, new idempotent nullnorms on bounded lattices have been also obtained.

Keywords

Lattice-based sum, Bounded lattice, Nullnorm, Idempotent nullnorm, T-norm, T-conorm.

1. INTRODUCTION

Nullnorm operators on the unit interval are special aggregation operators that have proven to be useful in many fields like expert systems, neural networks, fuzzy quantifiers, and fuzzy logics, see e.g. [13] and the references therein. They were originally introduced in [2, 17] as generalizations of triangular norms (t-norm for short) and triangular conorms (t-conorm for short) with the zero element a laying anywhere in the unit interval and have to satisfy some additional conditions. Nullnorms on the unit interval have been also studied in the papers [8, 18, 19] and many others.

In [15] nullnorms have been studied on bounded lattices where the existence of nullnorms with the zero element a laying anywhere in arbitrary bounded lattice L has been proven with underling t-norms and t-conorms on L. As a by-product, the existence of the smallest nullnorm and of the greatest nullnorm has been shown. Moreover, in [14], the existence of idempotent nullnorms on a distributive bounded lattice L has been also shown for any zero element $a \in L \backslash \{\bot, \top\}$. Recently an increasing interest of nullnorms on bounded lattices can be observed, see e.g. [4-6, 12] and many others.

In this paper, new methods for constructing nullnorms on bounded lattices are proposed. The proposed construction methods are based on the lattice-based sum of lattices that has been recently introduced by El-Zekey et al. (see [9]), for building new (bounded) lattices from fixed ones indexed by a (finite) lattice-ordered index set. Subsequently, the new construction methods are applied for building several new families of nullnorms on bounded lattices. As a by-product, lattice-based sum constructions of t-norms and t-conorms on bounded lattices have been obtained. Furthermore, new idempotent nullnorms on bounded lattices, different from the ones given in [6], have been also obtained. We point out that, unlike [6], in our construction method of the idempotent nullnorms, the underlying lattices need not to be distributive.

This paper is organized as follow. In Section 1, some basic notions are recalled. In Section 2, the basic results concerning the lattice-based sum of bounded lattices have been shortly recalled. In Section 3, lattice-based sum construction methods of nullnorms on bounded lattices have been developed. In Section 4, others lattice-based sum constructions of nullnorms leading to new idempotent nullnorms on bounded lattices have been also investigated. In Section 5, the results, from Section 3 and 4, are applied for constructing several new nullnorms on bounded lattices. Finally, some concluding remarks are added.

Definition 1. ([1, 7]) A *bounded lattice* $(L, \leq, \bot, \top)$ is a lattice which has the top and bottom elements, which are written as $\top$ and $\bot$, respectively, i.e. there exist two elements $\top, \bot \in L$ such that $\bot \leq x \leq \top$, for all $x \in L$.

Definition 2. ([3, 16]) An operation $T: L^2 \to L$ $(S: L^2 \to L)$ is called a t-norm (t-conorm) if it is commutative, associative, increasing with respect to both variables and has a neutral element $e = \top$ $(e = \bot)$.

Note that, the t-norm and the t-conorm are dual of each other. Therefore, by duality, the general properties of t-norms can be translated to their dual t-conorms.

Example 1. For any bounded lattice $(L, \leq, \bot, \top)$, there exist at least two t-norms and two t-conorms, as follows

i. The minimum t-norm $T_M^L: L^2 \to L$, $T_M^L(x, y) = x \wedge y$.
ii. The drastic product t-norm $T_D^L: L^2 \to L$, $T_D^L(x, y) = \begin{cases} x \wedge y & if\ \top \in \{x, y\}, \\ \bot & otherwise. \end{cases}$
iii. The maximum t-conorm $S_M^L: L^2 \to L$, $S_M^L(x, y) = x \vee y$.
iv. The drastic sum t-conorm $S_D^L: L^2 \to L$, $S_D^L(x, y) = \begin{cases} x \vee y & if\ \bot \in \{x, y\}, \\ \top & otherwise. \end{cases}$

Definition 3. ([15]) Let $(L, \leq, \bot, \top)$ be a bounded lattice. A commutative, associative, non-decreasing in each variable function $V: L^2 \to L$ is called a nullnorm if there is an element $a \in L$ such that $V(x, \bot) = x$ for all $x \leq a$, $V(x, \top) = x$ for all $x \geq a$.
It can be easily verified that $V(x, a) = a$ for all $x \in L$, i.e. $a \in L$ is the zero element of V.

Definition 4. ([1, 7]) Let $(L, \leq, \perp, \top)$ be a bounded lattice and $a \in L$. The downset of a denoted $\downarrow a$ and the upset of a denoted $\uparrow a$ are given by

$$\downarrow a = \{x \in L | x \leq a\}$$
$$\uparrow a = \{x \in L | x \geq a\}$$

2. LATTICE-BASED SUM OF BOUNDED LATTICES

In this section we briefly recall the lattice-based sum construction of lattice ordered sets introduced in [9] for building new lattice-ordered structures from the fixed ones indexed by a lattice-ordered index set.

In the sequel, $(\Lambda, \sqsubseteq)$ denotes a finite lattice-ordered index set. The top and bottom elements of $(\Lambda, \sqsubseteq)$ will be denoted by $\top_\Lambda$ and $\perp_\Lambda$, respectively. Further, each summand $(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ is a *bounded lattice* has a top element $\top_\alpha$ and a bottom element $\perp_\alpha$ for each $\alpha \in \Lambda$. We will use the lowercase Latin letters such as "x", "y" and "z" to ranging over the elements of L_α, and the lowercase Greek letters such as "α", "β" and "δ" to ranging over the elements of Λ. If there exist $\beta, \delta \in \Lambda$ such that β is incomparable with δ, then we will write $\beta \parallel \delta$. If $\beta, \delta \in \Lambda$ such that $\beta \sqsubseteq \delta$ but $\beta \neq \delta$, then we will write $\beta \sqsubset \delta$. The number of elements (the cardinality) of a set L will be denoted by $|L|$.

Definition 5. ([9]) Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$. The *Λ-sum family* is a family of bounded lattices $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)\}_{\alpha \in \Lambda}$ that satisfy for all $\alpha, \beta \in \Lambda$ with $\alpha \neq \beta$ the sets L_α and L_β are either disjoint or satisfy one of the following two conditions:

i. $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$ with $\alpha \sqsubset \beta$, where $x_{\alpha\beta}$ is both the top element of L_α and the bottom element of L_β, and where for each $\varepsilon \in \Lambda$ with $\alpha \sqsubset \varepsilon \sqsubset \beta$ we have $L_\varepsilon = \{x_{\alpha\beta}\}$, also for all $\delta, \gamma \in \Lambda$ with $\delta \parallel \gamma$, $\delta \sqsubset \beta$ and $\alpha \sqsubset \gamma$ we have $L_\delta = \{y_{\delta\gamma}\}$ or $L_\gamma = \{z_{\delta\gamma}\}$ where $y_{\delta\gamma}$ is the top element of $L_{\inf\{\delta,\gamma\}}$ and $z_{\delta\gamma}$ is the bottom element of $L_{\sup\{\delta,\gamma\}}$.
ii. $1 \leq |L_\alpha \cap L_\beta| \leq 2$ with $\alpha \parallel \beta$, and for each $x_{\alpha\beta} = L_\alpha \cap L_\beta$, $x_{\alpha\beta}$ is the top element of both L_α and L_β and the bottom element of $L_{\sup\{\alpha,\beta\}}$, or $x_{\alpha\beta}$ is the bottom element of both L_α and L_β and the top element of $L_{\inf\{\alpha,\beta\}}$.

Definition 6. ([9]) Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)\}_{\alpha \in \Lambda}$ be a Λ-sum family of bounded lattices. The lattice-based sum $\bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ is the set $L = \bigcup_{\alpha \in \Lambda} L_\alpha$ equipped with the order relation $\leq$ given by:

$x \leq y$ if and only if

$$\begin{cases} \exists \alpha \in \Lambda \text{ such that } x, y \in L_\alpha \textit{ and } x \leq_\alpha y \\ \textit{or} \\ \exists \alpha, \beta \in \Lambda \textit{ such that } (x, y) \in L_\alpha \times L_\beta \textit{ and } \alpha \sqsubset \beta \end{cases} \quad (1)$$

Theorem 1. ([9]) Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $\{(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)\}_{\alpha \in \Lambda}$ be a Λ-sum family of bounded lattices. Put $L = \bigcup_{\alpha \in \Lambda} L_\alpha$, then $(L, \leq, \perp, \top) = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ is a bounded lattice.

Note that, the partial order relation $\leq$ on the lattice L in Theorem 1 obtained by setting $x \leq y$ in L if and only if $x \wedge y = x$ coincides with the partial order relation given in (1). One obtains the same partial order relation from the given lattice by setting $x \leq y$ in L if and only if $x \vee y = y$.

Remark 1. We point out that, the lattice-ordered index set in [9] need not to be finite. But, for our purposes in this work, and from a practical point of view, from the very start, we assume that the index set is finite.

Note that, the strategy just described is focuses on the union of the carriers and an order consistent with both the order of the underlying bounded lattices and the order of the lattice-ordered index set (see Definition 6). Consequently, the order relation for elements from different summands is inherited from the lattice-ordered index set.

Example 2. Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ in Fig. 1. Then, the family associated with the structure in Fig. 2 is not a Λ-sum family because $L_\alpha \cap L_\beta = \{x_{\alpha\beta}\}$ with $x_{\alpha\beta} = \top_\alpha = \perp_\beta, \delta \sqsubset \beta, \alpha \sqsubset \gamma$ but neither $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ nor $L_\gamma = \{\perp_{\sup\{\delta,\gamma\}}\}$ violating the condition (i) in Definition 5. Note that, although the structure in Fig. 2 is a bounded lattice, its order relation isn't consistent with the order of the index set, since for $x \in L_\delta$ and $y \in L_\gamma$ we have $x \leq y$ but the only elements δ and γ in the index set associated with x and y, respectively, are incomparable elements in Λ. One of modifications of the family associated with the structure in Fig. 2 by putting $L_\delta = \{\top_{\inf\{\delta,\gamma\}}\}$ which produces the Λ-sum family of bounded lattices in Fig. 3. In this case, as x and y just described, we have $x \leq y, x \in L_\delta \cap L_{\perp_\Lambda}$ and $y \in L_\gamma$, and hence we have $\perp_\Lambda, \gamma \in \Lambda$ associated with x and y, respectively, such that $\perp_\Lambda \sqsubset \gamma$.

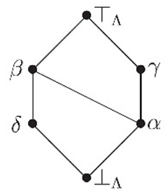

Fig 1. The lattice $(\Lambda, \sqsubseteq)$ of example 2

Note that, consequently from Definition 6 and Theorem 1, if the lattice-ordered index set is *linear*, then the lattice-based sum is reduced to the ordinal sum. For more details see [9].

We end this section by the following lemma, from [10], which is a direct consequence from Definition 6 and Theorem 1.

Lemma 1. ([10]) Let $(\Lambda, \sqsubseteq)$ be a finite lattice-ordered index set and let $L = \bigoplus_{\alpha \in \Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Assume that there exist $x_1, x_2 \in L$ such that there is no $\alpha \in \Lambda$ such that $\{x_1, x_2\} \subseteq L_\alpha$

i. If $x_1 < x_2$, then there exist $\alpha_1, \alpha_2 \in \Lambda$ such that $(x_1, x_2) \in L_{\alpha_1} \times L_{\alpha_2}$ with $\alpha_1 \sqsubset \alpha_2$ and for all $z_1 \in L_{\alpha_1}$ and for all $z_2 \in L_{\alpha_2}$ we have $z_1 \leq z_2$.
ii. If $x_1 \parallel x_2$, then for all $\alpha_1 \in I_{x_1}$ and $\alpha_2 \in I_{x_2}$ we have $\alpha_1 \parallel \alpha_2$ and for all $z_1 \in L_{\alpha_1} \backslash \{\perp_{\alpha_1}, \top_{\alpha_1}\}$ and for all $z_2 \in L_{\alpha_2} \backslash \{\perp_{\alpha_2}, \top_{\alpha_2}\}$ we have $z_1 \parallel z_2$.

Example 3. Consider the Λ-sum family of bounded lattices in Fig. 3. It is clear that, for all $x \in L_\alpha$ and $y \in L_\beta$ we have $x \leq y$ (since $\alpha \sqsubset \beta$). Further, for all $a \in L_\beta \backslash \{\perp_\beta, \top_\beta\}$ and $b \in L_\gamma \backslash \{\perp_\gamma, \top_\gamma\}$ we have $a \parallel b$ (since $\beta \parallel \gamma$).

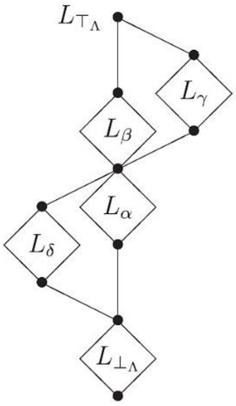

Fig 2. Not family

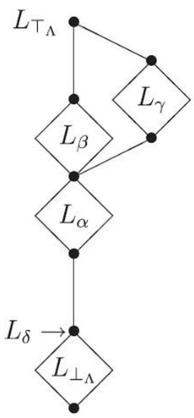

Fig 3. The Λ-sum family of example 2

3. LATTICE-BASED SUM CONSTRUCTION OF NULLNORMS ON BOUNDED LATTICES

In this section, based on the lattice-based sum of bounded lattices (see Section 2), we introduce a new method for constructing nullnorms on bounded lattices.

Remark 2. Under the assumption that each summand of the Λ-sum family is a bounded lattice, then for some lattice-ordered index set $(\Lambda, \sqsubseteq)$ and for any $\alpha \in \Lambda$ we have that for any t-norm T_α on L_α and for any t-conorm S_α on L_α,

$$T_\alpha(x,y) = x \wedge y, S_\alpha(x,y) = x \vee y.$$

when x or y are equal to one of the boundaries of L_α.

Theorem 2. Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and let $L = \bigoplus_{\alpha\in\Lambda} (L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$ be a lattice-based sum of bounded lattices. Let $a \in L$ with $a \in \{\perp_\alpha, \top_\alpha\}$ for some $\alpha \in \Lambda$ and $(T_\alpha)_{\alpha\in\Lambda}$ $((S_\alpha)_{\alpha\in\Lambda})$ be a family of t-norms (t-conorms) on the corresponding summands $(L_\alpha)_{\alpha\in\Lambda}$. Then the functions $V_\vee: L^2 \to L$ and $V_\wedge: L^2 \to L$ defined as follow

$$V_\vee(x,y) = \begin{cases} S_\alpha(x,y) & if\ x,y \in L_\alpha \cap\downarrow a, \\ T_\beta(x,y) & if\ x,y \in L_\beta \cap\uparrow a, \\ x \wedge y & if\ x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta, \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases} \tag{2}$$

$$V_\wedge(x,y) = \begin{cases} S_\alpha(x,y) & if\ x,y \in L_\alpha \cap\downarrow a, \\ T_\beta(x,y) & if\ x,y \in L_\beta \cap\uparrow a, \\ x \vee y & if\ x \in L_\alpha \cap\downarrow a, y \in L_\beta \cap\downarrow a, \alpha \neq \beta, \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases} \tag{3}$$

are nullnorms on L with zero element a.

Proof: The proof runs only for the operation $V_\vee$. The operation $V_\wedge$ has a similar proof.

First, we note that, for all $x, y \in L$ with $x, y \in\downarrow a$ and there is no $\alpha \in \Lambda$ such that $\{x,y\} \subseteq L_\alpha$ then we have $V_\vee(x,y) = (x \wedge a) \vee (y \wedge a) = x \vee y$. Also for all $x \in\uparrow a$ and for $y \parallel a$ or $y \in\downarrow a$, we have $V_\vee(x,y) = (x \wedge a) \vee (y \wedge a) = a \vee (y \wedge a) = a$, by absorption. We will always use this without mention.

It is necessary to check that the operation $V_\vee$ is well-defined. A problem can only arise if $(x,y) \in L_\alpha \times L_\beta$ with $x \in L_\alpha \cap L_\beta$ for some $\alpha, \beta \in \Lambda$ and say,

i. $x, y \in\downarrow a$,
   a) $\alpha \sqsubset \beta$,
      $$V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = y$$
      if we consider that $x, y \in L_\beta$, and
      $$V_\vee(x,y) = x \vee y = y$$
      if we consider that $x \in L_\alpha$ and $y \in L_\beta$. Thus producing the same result in both cases

   b) $\alpha \parallel \beta$, then either $x = \top_\alpha = \top_\beta$ and hence,
      $$V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = x \vee y = x$$
      or $x = \perp_\alpha = \perp_\beta$ and hence,
      $$V_\vee(x,y) = S_\beta(x,y) = x \vee_\beta y = x \vee y = y$$

ii. $x, y \in\uparrow a$. This case is dual to case (i) has a dual proof due to the duality between the t-norm and the t-conorm.

Now, we need to prove that $V_\vee$ is a nullnorm on L with zero element a.

It is easy to see the commutativity of $V_\vee$ due to the commutativity of the t-norm and the t-conorm defined on each summand and $\wedge$ and $\vee$ on L.

Zero element: We prove that a is the zero element of $V_\vee$. The proof is split into all the possible cases for some $x \in L$, as follows.

i. $x \in\downarrow a$,
   a) There exist some $\alpha \in \Lambda$ such that $\{x, a\} \subseteq L_\alpha$, then (from Remark 2) we have,
      $$V_\vee(x,a) = S_\alpha(x,a) = x \vee_\alpha a = a$$
   b) There is no $\alpha \in \Lambda$ such that $\{x, a\} \subseteq L_\alpha$, then,
      $$V_\vee(x,a) = x \vee a = a$$
ii. $x \in\uparrow a$. This case has a dual proof of case (i) due to the duality between the t-norm and the t-conorm.
iii. $x \parallel a$. Then directly from the definition of $V_\vee$ we have
      $$V_\vee(x,a) = a$$

Monotonicity: We prove that if $x \leq y$ in L, then for all $z \in L$, $V_\vee(x,z) \leq V_\vee(y,z)$. The proof is split into all the possible cases, as follows,

Case (1): Let $x, y \in\downarrow a$. Then we have the following subcases

Subcase 1(a): $z \in\downarrow a$,

i. There exist some $\alpha \in \Lambda$ such that $\{x,y\} \subseteq L_\alpha$. If $z \in L_\alpha$, then monotonicity holds trivially due to the monotonicity of S_α on L_α. If $z \notin L_\alpha$, then
$$V_\vee(x,z) = x \vee z \leq y \vee z = V_\vee(y,z)$$
ii. There is no $\alpha \in \Lambda$ such that $\{x,y\} \subseteq L_\alpha$.
   a) If x and z are in the same summand, we observe it by consider $\{x,z\} \subseteq L_\beta$ and $y \in L_\delta$ with $\beta \neq \delta$ for some $\beta, \delta \in \Lambda$, then from Lemma 1, we have either $\beta \sqsubset \delta$ or $\beta \parallel \delta$. If $\beta \sqsubset \delta$, then
$$\begin{aligned} V_\vee(x,z) &= S_\beta(x,z) \leq y \\ &= y \vee z = V_\vee(y,z) \end{aligned}$$
   If $\beta \parallel \delta$, then we have either $x \in \{\bot_\beta, \top_\beta\}$ and hence,
$$\begin{aligned} V_\vee(x,z) &= S_\beta(x,z) = x \vee z \leq y \vee z \\ &= V_\vee(y,z) \end{aligned}$$
   or $x \in L_\beta \backslash \{\bot_\beta, \top_\beta\}$, then necessarily $y = \top_\delta$ and hence,
$$\begin{aligned} V_\vee(x,z) &= S_\beta(x,z) \leq y \\ &= y \vee z = V_\vee(y,z) \end{aligned}$$
   b) If y and z are in the same summand, we observe it by consider $\{y,z\} \subseteq L_\alpha$ and $x \notin L_\alpha$ for some $\alpha \in \Lambda$, then
$$\begin{aligned} V_\vee(x,z) &= x \vee z \leq y \vee z \leq S_\alpha(y,z) \\ &= V_\vee(y,z) \end{aligned}$$
   c) All arguments are in different summands,
$$V_\vee(x,z) = x \vee z \leq y \vee z = V_\vee(y,z)$$

Subcase 1(b): $z \in\uparrow a$,
$$V_\vee(x,z) = a = V_\vee(y,z)$$

Subcase 1(c): $z \parallel a$,
$$\begin{aligned} V_\vee(x,z) &= (x \wedge a) \vee (z \wedge a) \\ &\leq (y \wedge a) \vee (z \wedge a) \\ &= V_\vee(y,z) \end{aligned}$$

Case (2): Let $x \in\uparrow a$, then $y \in\uparrow a$.

i. $z \in\downarrow a$,
$$V_\vee(x,z) = a = V_\vee(y,z)$$
ii. $z \in\uparrow a$. In this case, the proof is a dual proof of case (1) due to the duality between the t-norm and the t-conorm.
iii. $z \parallel a$,
$$V_\vee(x,z) = a = V_\vee(y,z)$$

Case (3): Let $x \in\downarrow a, y \in\uparrow a$.

i. $z \in\downarrow a$. In this case we have either x and z are in the same summand or x and z are in different summands. In both cases and due to the t-conorm and $\vee$ on L we have,
$$V_\vee(x,z) \leq a = V_\vee(y,z)$$
ii. $z \in\uparrow a$. Similarly, as in case (i) we have $V_\vee(y,z) \geq a$ and hence,
$$V_\vee(x,z) = a \leq V_\vee(y,z)$$
iii. $z \parallel a$,
$$V_\vee(x,z) = (x \wedge a) \vee (z \wedge a) \leq a = V_\vee(y,z)$$

Case (4): Let $x \in\downarrow a, y \parallel a$.

i. $z \in\downarrow a$,
   a) There exist some $\alpha \in \Lambda$ such that $\{x,z\} \subseteq L_\alpha$. Then we have either $y \wedge a \in L_\alpha$ or $y \wedge a \notin L_\alpha$. If $y \wedge a \in L_\alpha$, then necessarily $y \in \{\bot_\alpha, \top_\alpha\}$. In case that $y \wedge a = \top_\alpha$, then
$$\begin{aligned} V_\vee(x,z) &= S_\alpha(x,z) \leq \top_\alpha \\ &= y \wedge a = (y \wedge a) \vee z \\ &= V_\vee(y,z) \end{aligned}$$
   In case that $y \wedge a = \bot_\alpha$, then necessarily $x = \bot_\alpha$ and hence
$$\begin{aligned} V_\vee(x,z) &= S_\alpha(x,z) = x \vee z \\ &= z = (y \wedge a) \vee z \\ &= V_\vee(y,z) \end{aligned}$$
   If $y \wedge a \notin L_\alpha$, then $y \wedge a > u$ for all $u \in L_\alpha$ and hence,
$$\begin{aligned} V_\vee(x,z) &= S_\alpha(x,z) \leq y \wedge a \\ &= (y \wedge a) \vee z = V_\vee(y,z) \end{aligned}$$
   b) There is no $\alpha \in \Lambda$ such that $\{x,z\} \subseteq L_\alpha$, then
$$\begin{aligned} V_\vee(x,z) &= (x \wedge a) \vee (z \wedge a) \\ &\leq (y \wedge a) \vee (z \wedge a) \\ &= V_\vee(y,z) \end{aligned}$$
ii. $z \in\uparrow a$,
$$V_\vee(x,z) = a = V_\vee(y,z)$$
iii. $z \parallel a$. This case is similar to subcase 1(c) has a similar proof.

Case (5): Let $x \parallel a, y \in\uparrow a$.

i. $z \in\downarrow a$,
$$V_\vee(x,z) = (x \wedge a) \vee (z \wedge a) \leq a = V_\vee(y,z)$$
ii. $z \in\uparrow a$. In similar way of case 3(ii) we have $V_\vee(y,z) \geq a$ and hence,
$$V_\vee(x,z) = a \leq V_\vee(y,z)$$
iii. $z \parallel a$,
$$V_\vee(x,z) = (x \wedge a) \vee (z \wedge a) \leq a = V_\vee(y,z)$$

Case (6): Let $x \parallel a, y \parallel a$.

i. $z \in\downarrow a$. This case is similar to case 4(iii) has a similar proof.
ii. $z \in\uparrow a$,
$$V_\vee(x,z) = a = V_\vee(y,z)$$
iii. $z \parallel a$,
$$\begin{aligned} V_\vee(x,z) &= (x \wedge a) \vee (z \wedge a) \\ &\leq (y \wedge a) \vee (z \wedge a) = V_\vee(y,z) \end{aligned}$$

Associativity: We prove that $V_\vee\left(V_\vee(x,y),z\right) = V_\vee\big(x, V_\vee(y,z)\big)$ for all $x, y, z \in L$. Again, the proof is split into all possible cases by considering the relationship between the arguments x, y, z and a, as follows.

Case (1): All arguments are from $\downarrow a$.

i. There exist some $\alpha \in \Lambda$ such that $\{x,y,z\} \subseteq L_\alpha$. In this case associativity holds trivially due to the associativity of S_α on L_α.
ii. All arguments are from different summands,
$$\begin{aligned} V_\vee\left(V_\vee(x,y),z\right) &= V_\vee\left(x \vee y, z\right) \\ &= x \vee y \vee z \\ &= V_\vee\left(x, y \vee z\right) \\ &= V_\vee\big(x, V_\vee(y,z)\big) \end{aligned}$$

In this case, we must note that, if $x \vee y$ and z are in the same summand, then necessarily $x \vee y$ is equal to one of the boundaries of this summand and hence (from Remark 2) we have $V_\vee(x \vee y, z) = x \vee y \vee z$.

iii. Exactly two arguments are from the same summand. We observe it by considering the following cases.

a) There exist some $\alpha \in \Lambda$ such that $\{x, y\} \subseteq L_\alpha$ and $z \notin L_\alpha$. If x or y is equal to one of the boundaries of L_α then (from Remark 2) associativity holds trivially due to the associativity of $\vee$ on L. Therefore, we assume that $x, y \in L_\alpha \backslash \{\perp_\alpha, \top_\alpha\}$, then we compare z with x and y, as follows

If $x > z$ or $y > z$, then
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee(S_\alpha(x, y), z) \\ &= S_\alpha(x, y) \vee z \\ &= S_\alpha(x, y) \\ &= S_\alpha(x, y \vee z) \\ &= V_\vee(x, V_\vee(y, z)) \end{aligned}$$
If $x < z$ or $y < z$, then
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee(S_\alpha(x, y), z) \\ &= S_\alpha(x, y) \vee z \\ &= z \\ &= y \vee z = V_\vee(y, z) \\ &= x \vee V_\vee(y, z) \\ &= V_\vee(x, V_\vee(y, z)) \end{aligned}$$
If $x \parallel z$ or $y \parallel z$, then
$$x \vee z = y \vee z = S_\alpha(x, y) \vee z$$
and hence,
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee(S_\alpha(x, y), z) \\ &= S_\alpha(x, y) \vee z \\ &= y \vee z = V_\vee(y, z) \\ &= x \vee V_\vee(y, z) \\ &= V_\vee(x, V_\vee(y, z)) \end{aligned}$$

b) There exist some $\beta \in \Lambda$ such that $\{x, z\} \subseteq L_\beta$ and $y \notin L_\beta$. This case is similar to case (a) has a similar proof.

c) There exist some $\delta \in \Lambda$ such that $\{y, z\} \subseteq L_\delta$ and $x \notin L_\delta$. This case is similar to case (a) has a similar proof.

Case (2): All arguments are from $\uparrow a$. This case has a dual proof of case (1) due to the duality between t-norm and t-conorm.

Case (3): All arguments are incomparable with a,
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ &= V_\vee\big(x, (y \wedge a) \vee (z \wedge a)\big) \\ &= V_\vee\big(x, V_\vee(y, z)\big) \end{aligned}$$

Case (4): Exactly two arguments are from $\downarrow a$.

i. $x, y \in\downarrow a, z > a$. In this case we have either x and y are in the same summand or x and y are from different summands. In both cases, we have $V_\vee(x, y) \leq a$ and hence,
$$V_\vee(V_\vee(x, y), z) = a = V_\vee(x, a) = V_\vee(x, V_\vee(y, z))$$

ii. $x, y \in\downarrow a, z \parallel a$. Then from the fact that $z \wedge a < a$, the associativity holds by a proof exactly similar to case (1) but with $x, y \in\downarrow a$ and $z \wedge a < a$.

iii. $x, z \in\downarrow a, y > a$,
$$V_\vee(V_\vee(x, y), z) = V_\vee(a, z) = a$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

iv. $x, z \in\downarrow a, y \parallel a$. This case is similar to case 4(ii) has a similar proof.

v. $y, z \in\downarrow a, x > a$. This case is similar to case 4(i) has a similar proof.

vi. $y, z \in\downarrow a, x \parallel a$. This case is similar to case 4(ii) has a similar proof.

Case (5): Exactly two arguments are from $\uparrow a$.

i. $x, y \in\uparrow a, z < a$. In this case we have either x and y are in the same summand or x and y are in different summands. In both cases, we have $V_\vee(x, y) \geq a$ and hence
$$V_\vee(V_\vee(x, y), z) = a$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

ii. $x, y \in\uparrow a, z \parallel a$. This case is similar to case 5(i) has a similar proof.

iii. $x, z \in\uparrow a, y < a$.
$$V_\vee(V_\vee(x, y), z) = V_\vee(a, z) = a$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

iv. $x, z \in\uparrow a, y \parallel a$. This case is similar to case 5(iii) has a similar proof.

v. $y, z \in\uparrow a, x < a$. In this case we have either y and z are in the same summand or y and z are in different summands. In both cases, we have $V_\vee(y, z) \geq a$ and hence
$$V_\vee(V_\vee(x, y), z) = V_\vee(x, a) = a$$
$$V_\vee(x, V_\vee(y, z)) = a$$

vi. $y, z \in\uparrow a, x \parallel a$. This case is similar to case 5(v) has a similar proof.

Case (6): Exactly two arguments are incomparable with a.

i. $x \parallel a, y \parallel a, z \in\downarrow a$.
In this case we have $x \wedge a < a$ and $y \wedge a < a$ with $x \wedge a$ and $y \wedge a$ are on the boundaries and hence (from Remark 2) we have
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ V_\vee(x, V_\vee(y, z)) &= V_\vee\big(x, (y \wedge a) \vee (z \wedge a)\big) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \end{aligned}$$

ii. $x \parallel a, y \parallel a, z \in\uparrow a$.
$$\begin{aligned} V_\vee(V_\vee(x, y), z) &= V_\vee\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) \\ &= a \end{aligned}$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

iii. $x \parallel a, z \parallel a, y \in\downarrow a$. This case is similar to case 6(i) has a similar proof.

iv. $x \parallel a, z \parallel a, y \in\uparrow a$,
$$V_\vee(V_\vee(x, y), z) = V_\vee(a, z) = a$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

v. $y \parallel a, z \parallel a, x \in\downarrow a$. This case is similar to case 6(i) has a similar proof.

vi. $y \parallel a, z \parallel a, x \in\uparrow a$. This case is similar to case 6(iv) has a similar proof.

For other possibilities we distinguish the following cases

i. $x \in\downarrow a, y \in\uparrow a, z \parallel a$,
$$V_\vee(V_\vee(x, y), z) = V_\vee(a, z) = a$$
$$V_\vee(x, V_\vee(y, z)) = V_\vee(x, a) = a$$

ii. $x \in\downarrow a, y \parallel a, z \in\uparrow a$,
$$V_\vee(V_\vee(x, y), z) = V_\vee\Big(\big((x \wedge a) \vee (y \wedge a)\big), z\Big) = a$$

$$V_\vee(x, V_\vee(y,z)) = V_\vee(x,a) = a$$

iii. $x \in \uparrow a, y \in \downarrow a, z \parallel a,$
$$V_\vee(V_\vee(x,y),z) = V_\vee(a,z) = a$$
$$V_\vee(x, V_\vee(y,z)) = V_\vee(x,(y \wedge a) \vee (z \wedge a)) = a$$

iv. $x \in \uparrow a, y \parallel a, z \downarrow a$. This case is similar to case (iii) has a similar proof.

v. $x \parallel a, y \downarrow a, z \in \uparrow a$. This case is similar to case (ii) has a similar proof.

vi. $x \parallel a, y \uparrow a, z \downarrow a$. This case is similar to case (i) has a similar proof.

In case of $a = \top$ we obtain t-conorms and in case of $a = \bot$ we obtain t-norms. Consequently, we get, as a corollary, the following lattice-based sum constructions of t-norms and t-conorms obtained by El-Zekey (see [10]) in a more general setting where the lattice-ordered index set need not to be finite and so-called t-subnorms (t-subconorms) can be used (with a little restriction) instead of t-norms (t-conorms) as summands in the lattice-based sum construction of t-norms (t-conorms).

Corollary 1. With all the assumptions of Theorem 2 the nullnorm functions $V_\vee$ and $V_\wedge$ as defined in equations (2) and (3), respectively, satisfy the following:

i. If $a = \bot$, then
$$V_\vee(x,y) = V_\wedge(x,y) = \begin{cases} T_\alpha(x,y) & if\ (x,y) \in L_\alpha \times L_\alpha, \\ x \wedge y & otherwise, \end{cases}$$
is a t-norm on L.

ii. If $a = \top$, then
$$V_\vee(x,y) = V_\wedge(x,y) = \begin{cases} S_\alpha(x,y) & if\ (x,y) \in L_\alpha \times L_\alpha, \\ x \vee y & otherwise, \end{cases}$$
is a t-conorm on L.

4. IDEMPOTENT NULLNORMS ON BOUNDED LATTICES

In this section, others lattice-based sum construction methods of nullnorms leading to new idempotent nullnorms on bounded lattices are investigated.

Theorem 3. Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and let $L = \bigcup_{\alpha \in \Lambda} L_\alpha$. Let $a \in L$ with $a \in \{\bot_\alpha, \top_\alpha\}$ for some $\alpha \in \Lambda$ and $(T_\alpha)_{\alpha \in \Lambda}$ $((S_\alpha)_{\alpha \in \Lambda})$ be a family of t-norms (t-conorms) on the corresponding summands $(L_\alpha)_{\alpha \in \Lambda}$. Then the functions $V_\vee^I: L^2 \to L$ and $V_\wedge^I: L^2 \to L$ defined as follow

$$V_\vee^I(x,y) = \begin{cases} S_\alpha(x,y) & if\ x,y \in L_\alpha \cap \downarrow a, \\ T_\beta(x,y) & if\ x,y \in L_\beta \cap \uparrow a, \\ x \wedge y & if\ (x \in L_\alpha \cap \uparrow a, y \in L_\beta \cap \uparrow a, \alpha \neq \beta)\ or\ (x = y \parallel a), \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases} \tag{4}$$

and

$$V_\wedge^I(x,y) = \begin{cases} S_\alpha(x,y) & if\ x,y \in L_\alpha \cap \downarrow a, \\ T_\beta(x,y) & if\ x,y \in L_\beta \cap \uparrow a, \\ x \vee y & if\ (x \in L_\alpha \cap \downarrow a, y \in L_\beta \cap \downarrow a, \alpha \neq \beta)\ or\ (x = y \parallel a), \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases} \tag{5}$$

are nullnorms on L with zero element a.

Proof: The proof runs only for the operation $V_\vee^I$. The operation $V_\wedge^I$ has a similar proof.

The commutativity, the monotonicity and the fact that a is the zero element of $V_\vee^I$ have exactly the same proof as the corresponding one from Theorem 2. It is only remaining to see the associativity of $V_\vee^I$.

Associativity: We prove that $V_\vee^I(V_\vee^I(x,y),z) = V_\vee^I(x, V_\vee^I(y,z))$ for all $x,y,z \in L$.

Associativity of $V_\vee^I$ is preserved in all cases by exactly the same proof of the corresponding cases from Theorem 2, but only one, namely if at least two equal arguments are incomparable with a. Therefore, we assume that $x = y \parallel a$ and distinguish the following cases

Case (1): $z \parallel a$ with $z \neq x$ (equivalent to $z \neq y$)

$$\begin{aligned} V_\vee^I(V_\vee^I(x,y),z) &= V_\vee^I(x \wedge y, z) \\ &= V_\vee^I(x,z) \\ &= (x \wedge a) \vee (z \wedge a) \\ V_\vee^I(x, V_\vee^I(y,z)) &= V_\vee^I(x,(y \wedge a) \vee (z \wedge a)) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ &= (x \wedge a) \vee (z \wedge a) \end{aligned}$$

Case (2): $z \in \downarrow a$,

$$\begin{aligned} V_\vee^I(V_\vee^I(x,y),z) &= V_\vee^I(x \wedge y, z) \\ &= V_\vee^I(x,z) \\ &= (x \wedge a) \vee (z \wedge a) \\ V_\vee^I(x, V_\vee^I(y,z)) &= V_\vee^I(x,(y \wedge a) \vee (z \wedge a)) \\ &= (x \wedge a) \vee (y \wedge a) \vee (z \wedge a) \\ &= (x \wedge a) \vee (z \wedge a) \end{aligned}$$

Case (3): $z \in \uparrow a$,

$$\begin{aligned} V_\vee^I(V_\vee^I(x,y),z) &= V_\vee^I(x \wedge y, z) \\ &= V_\vee^I(x,z) = a \\ V_\vee^I(x, V_\vee^I(y,z)) &= V_\vee^I(x,(y \wedge a) \vee (z \wedge a)) \\ &= V_\vee^I(x,(y \wedge a) \vee a) \\ &= V_\vee^I(x,a) = a \end{aligned}$$

All other cases can be shown in similar way.

Corollary 2. If we put $T_\alpha = T_M^L$ and $S_\alpha = S_M^L$ on L_α for all $\alpha \in \Lambda$ in $V_\vee^I$ and $V_\wedge^I$ in equations (4) and (5) in Theorem 3, then the following functions are idempotent nullnorms on L with zero element a.

$$V_\vee^I(x,y) = \begin{cases} x \wedge y & if\ (x,y \in \uparrow a)\ or\ (x = y \parallel a), \\ (x \wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

and

$$V_\wedge^I(x,y) = \begin{cases} x \vee y & if\ (x,y \in \downarrow a)\ or\ (x = y \parallel a), \\ (x \vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Remark 3. The zero element a of the nullnorms $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ was restricted to be one of the boundaries of some summand of L. If a is inside some summand, then $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ may not work to construct nullnorms on L, for example, if we consider a lattice index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices L and there exist some $\alpha \in \Lambda$ such that $\{x,y,a\} \subseteq L_\alpha$ with $\bot_\alpha < x < a < y < \top_\alpha$ and $T_\alpha = T_D^L$, $S_\alpha = S_D^L$ then from Theorem 2 and Theorem 3 we have

$$V_\vee(x,a) = V_\wedge(x,a) = V_\vee^I(x,a) = V_\wedge^I(x,a) = S_\alpha(x,a) = S_D^L(x,a) = \top_\alpha \neq a$$

$$V_\vee(y,a) = V_\wedge(y,a) = V_\vee^I(y,a) = V_\wedge^I(y,a) = T_\alpha(y,a) = T_D^L(y,a) = \perp_\alpha \neq a$$

Violating the zero element property of the nullnorm operator. However, the functions $V_\vee, V_\wedge, V_\vee^I$ and $V_\wedge^I$ are still nullnorms on L in case that a is inside some summand if and only if the t-norm and the t-conorm defined on this summand are fixed to be the minimum T_M^L and the maximum S_M^L, respectively.

5. Applications

In this section, the obtained results are applied for building several new nullnorm operations on bounded lattices.

Corollary 3. Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha\in\Lambda}(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$. If we put $S_\alpha = S_D^L$ and $T_\alpha = T_D^L$ for all $\alpha \in \Lambda$ in $V_\vee$ and $V_\wedge$ in Theorem 2, then we obtain the following nullnorms

$$V_\vee^D(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in (L_\beta \cap\uparrow a)\backslash\{\top_\beta\}, \\ x \wedge y & if\ x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta, \\ (x\wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

$$V_\wedge^D(x,y) = \begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in (L_\beta \cap\uparrow a)\backslash\{\top_\beta\}, \\ x \vee y & if\ x \in L_\alpha \cap\downarrow a, y \in L_\beta \cap\downarrow a, \alpha \neq \beta, \\ (x\vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Corollary 4. Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha\in\Lambda}(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$. If we put $S_\alpha = S_D^L$ and $T_\alpha = T_D^L$ for all $\alpha \in \Lambda$ in $V_\vee^I$ and $V_\wedge^I$ in Theorem 3, then we obtain the following nullnorms

$$V_\vee^d(x,y) =$$

$$\begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in (L_\beta \cap\uparrow a)\backslash\{\top_\beta\}, \\ x \wedge y & if\ \left(x \in L_\alpha \cap\uparrow a, y \in L_\beta \cap\uparrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x\wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

$$V_\wedge^d(x,y) =$$

$$\begin{cases} \top_\alpha & if\ x,y \in (L_\alpha \cap\downarrow a)\backslash\{\perp_\alpha\}, \\ \perp_\beta & if\ x,y \in (L_\beta \cap\uparrow a)\backslash\{\top_\beta\}, \\ x \vee y & if\ \left(x \in L_\alpha \cap\downarrow a, y \in L_\beta \cap\downarrow a, \alpha \neq \beta\right)\ or\ (x = y \parallel a), \\ (x\vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Corollary 5. Consider a finite lattice-ordered index set $(\Lambda, \sqsubseteq)$ and a lattice-based sum of bounded lattices $L = \bigoplus_{\alpha\in\Lambda}(L_\alpha, \leq_\alpha, \perp_\alpha, \top_\alpha)$. If we put $S_\alpha = S_M^L$ and $T_\alpha = T_M^L$ for all $\alpha \in \Lambda$ in $V_\vee$ and $V_\wedge$ in Theorem 2, then we obtain the following nullnorms

$$V_\vee^M(x,y) = \begin{cases} x \wedge y & if x,y \in\uparrow a, \\ (x\wedge a) \vee (y \wedge a) & otherwise. \end{cases}$$

and

$$V_\wedge^M(x,y) = \begin{cases} x \vee y & if\ x,y \in\downarrow a, \\ (x\vee a) \wedge (y \vee a) & otherwise. \end{cases}$$

Example 4. Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ shown in Fig. 4 and the lattice-based sum of bounded lattices L shown in Fig. 5 where $L_{\perp_\Lambda} = \{0\}$, $L_\alpha = \{x,y,z,t,g\}$, $L_\beta = \{g\}$, $L_\delta = \{a,b,c,d,e,f\}$, and $L_{\top_\Lambda} = \{g,h,m,n,1\}$. Let $T_{\top_\Lambda}$ be the t-norm defined on $L_{\top_\Lambda}$ whose values are written in Table 1 and S_δ be the t-conorm on L_δ whose values are written in Table 2. Then the functions $V_\vee$ and $V_\wedge$ whose values are written in Table 3 and Table 4 are, respectively, nullnorms on L with zero element a which are constructed using equations (2) and (3), respectively.

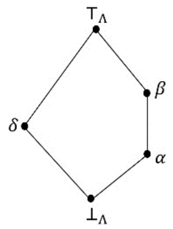

Fig 4. The lattice $(\Lambda, \sqsubseteq)$ of example 4

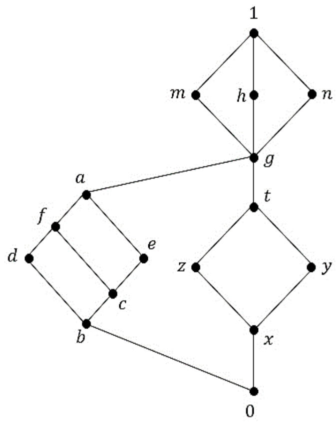

Fig 5. The lattice L of example 4

Table 1. The t-norm $T_{\top_\Lambda}$ on $L_{\top_\Lambda}$

$T_{\top_\Lambda}$	g	h	m	n	1
g	g	g	g	g	g
h	g	h	g	g	h
m	g	g	g	g	m
n	g	g	g	g	n
1	g	h	m	n	1

Table 2. The t-conorm S_δ on L_δ

S_δ	b	c	d	e	f	a
b	b	c	d	e	f	a
c	c	f	f	a	f	a
d	d	f	f	a	f	a
e	e	a	a	a	a	a
f	f	f	f	a	f	a
a	a	a	a	a	a	a

Table 3. The nullnorm $V_\vee$ on L of example 4

$V_\vee$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
x	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
y	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
z	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
t	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
g	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	g
b	b	b	b	b	b	a	b	c	d	e	f	a	a	a	a	a
c	c	c	c	c	c	a	c	f	f	a	f	a	a	a	a	a
d	d	d	d	d	d	a	d	f	f	a	f	a	a	a	a	a
e	e	e	e	e	e	a	e	a	a	a	a	a	a	a	a	a
f	f	f	f	f	f	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	a	a	a	a	g	a	a	a	a	a	a	h	g	g	h
m	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	m
n	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	n
1	a	a	a	a	a	g	a	a	a	a	a	a	h	m	n	1

Table 4. The nullnorm $V_\wedge$ on L of example 4

$V_\wedge$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
x	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
y	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
z	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
t	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
g	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
b	b	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
c	c	a	a	a	a	a	c	f	f	a	f	a	a	a	a	a
d	d	a	a	a	a	a	d	f	f	a	f	a	a	a	a	a
e	e	a	a	a	a	a	e	a	a	a	a	a	a	a	a	a
f	f	a	a	a	a	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	g	g	g	g	g	a	a	a	a	a	a	h	g	g	h
m	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	m
n	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	n
1	a	g	g	g	g	g	a	a	a	a	a	a	h	m	n	1

Example 5. Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ and the lattice-based sum of bounded lattices L of example 4. With the same data of example 4, then the functions $V_\vee^I$ and $V_\wedge^I$ whose values are written in Table 5 and Table 6 are, respectively, nullnorms on L with zero element a which are constructed using equations (4) and (5), respectively.

Table 5. The nullnorm $V_\vee^I$ on L of example 5

$V_\vee^I$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	0	0	0	0	a	b	c	d	e	f	a	a	a	a	a
x	0	x	0	0	0	a	b	c	d	e	f	a	a	a	a	a
y	0	0	y	0	0	a	b	c	d	e	f	a	a	a	a	a
z	0	0	0	z	0	a	b	c	d	e	f	a	a	a	a	a
t	0	0	0	0	t	a	b	c	d	e	f	a	a	a	a	a
g	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	g
b	b	b	b	b	b	a	b	c	d	e	f	a	a	a	a	a
c	c	c	c	c	c	a	c	f	f	a	f	a	a	a	a	a
d	d	d	d	d	d	a	d	f	f	a	f	a	a	a	a	a
e	e	e	e	e	e	a	e	a	a	a	a	a	a	a	a	a
f	f	f	f	f	f	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	a	a	a	a	g	a	a	a	a	a	a	h	g	g	h
m	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	m
n	a	a	a	a	a	g	a	a	a	a	a	a	g	g	g	n
1	a	a	a	a	a	g	a	a	a	a	a	a	h	m	n	1

Table 6. The nullnorm $V_\wedge^I$ on L of example 5

$V_\wedge^I$	0	x	y	z	t	g	b	c	d	e	f	a	h	m	n	1
0	0	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
x	a	x	g	g	g	g	a	a	a	a	a	a	g	g	g	g
y	a	g	y	g	g	g	a	a	a	a	a	a	g	g	g	g
z	a	g	g	z	g	g	a	a	a	a	a	a	g	g	g	g
t	a	g	g	g	t	g	a	a	a	a	a	a	g	g	g	g
g	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	g
b	b	a	a	a	a	a	b	c	d	e	f	a	a	a	a	a
c	c	a	a	a	a	a	c	f	f	a	f	a	a	a	a	a
d	d	a	a	a	a	a	d	f	f	a	f	a	a	a	a	a
e	e	a	a	a	a	a	e	a	a	a	a	a	a	a	a	a
f	f	a	a	a	a	a	f	f	f	a	f	a	a	a	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
h	a	g	g	g	g	g	a	a	a	a	a	a	h	g	g	h
m	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	m
n	a	g	g	g	g	g	a	a	a	a	a	a	g	g	g	n
1	a	g	g	g	g	g	a	a	a	a	a	a	h	m	n	1

Example 6. Consider the lattice-ordered index set $(\Lambda, \sqsubseteq)$ of example 4 and its lattice-based sum of bounded lattices L in Fig. 6. Let $S_{\perp_\Lambda} = S_D^L$, then the functions $V_\vee$ and $V_\wedge$ whose values are written in Table 7 and Table 8, respectively, are nullnorms on L with zero element a which are constructed using equations (2) and (3), respectively. Note that, a is inside L_δ, then according to Remark 3, the t-norm T_δ and the t-conorm S_δ are considered to be the minimum T_M^L and the maximum S_M^L, respectively.

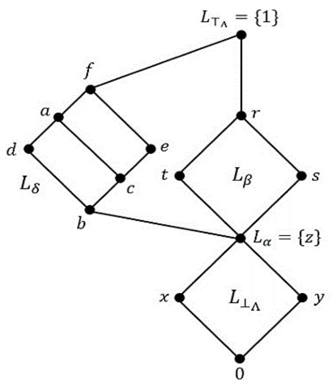

Fig 6. The lattice L of example 6

Table 7. The nullnorm $V_\vee$ on L of example 6

$V_\vee$	0	x	y	z	t	s	r	a	b	c	d	e	f	1
0	0	x	y	z	z	z	z	a	b	c	d	c	a	a
x	x	z	z	z	z	z	z	a	b	c	d	c	a	a
y	y	z	z	z	z	z	z	a	b	c	d	c	a	a
z	z	z	z	z	z	z	z	a	b	c	d	c	a	a
t	z	z	z	z	z	z	z	a	b	c	d	c	a	a
s	z	z	z	z	z	z	z	a	b	c	d	c	a	a
r	z	z	z	z	z	z	z	a	b	c	d	c	a	a
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	b	b	b	a	b	c	d	c	a	a
c	c	c	c	c	c	c	c	a	c	c	a	c	a	a
d	d	d	d	d	d	d	d	a	d	a	d	a	a	a
e	c	c	c	c	c	c	c	a	c	c	a	c	a	a
f	a	a	a	a	a	a	a	a	a	a	a	a	f	f
1	a	a	a	a	a	a	a	a	a	a	a	a	f	1

Table 8. The nullnorm $V_\wedge$ on L of example 6

$V_\wedge$	0	x	y	z	t	s	r	a	b	c	d	e	f	1
0	0	x	y	z	a	a	a	a	b	c	d	a	a	a
x	x	z	z	z	a	a	a	a	b	c	d	a	a	a
y	y	z	z	z	a	a	a	a	b	c	d	a	a	a
z	z	z	z	z	a	a	a	a	b	c	d	a	a	a
t	a	a	a	a	1	1	1	a	a	a	a	f	f	1
s	a	a	a	a	1	1	1	a	a	a	a	f	f	1
r	a	a	a	a	1	1	1	a	a	a	a	f	f	1
a	a	a	a	a	a	a	a	a	a	a	a	a	a	a
b	b	b	b	b	a	a	a	a	b	c	d	a	a	a
c	c	c	c	c	a	a	a	a	c	c	a	a	a	a
d	d	d	d	d	a	a	a	a	d	a	d	a	a	a
e	a	a	a	a	f	f	f	a	a	a	a	f	f	f
f	a	a	a	a	f	f	f	a	a	a	a	f	f	f
1	a	a	a	a	1	1	1	a	a	a	a	f	f	1

6. CONCLUDING REMARKS

In this paper, based on the lattice-based sum scheme that has been recently introduced by El-Zekey et al. (see [9]); new methods for constructing nullnorms on bounded lattices which are a lattice-based sum of their summand sublattices are developed. Subsequently, the obtained results are applied for building several new nullnorm operations on bounded lattices. As a by-product, the lattice-based sum constructions of t-norms and t-conorms obtained by El-Zekey (see [10]) are obtained in a more general setting where the lattice-ordered index set need not to be finite and so-called t-subnorms (t-subconorms) can be used (with a little restriction) instead of t-norms (t-conorms) as summands. Furthermore, new idempotent nullnorms on bounded lattices, different from the ones given in [6], have been also obtained. It is pointed out that, unlike [6], in our construction of the idempotent nullnorms, the underlying lattices need not to be distributive. We remark that lattice-based sum constructions of other aggregation functions on bounded lattices could also be taken into account (compare also, e.g. [10, 11]).

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